

Resumindo:

Eq. Laplace Coord. Esféricas: $\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$

Simetria de rotações em torno do eixo Z: $\rightarrow \frac{\partial V}{\partial \phi} = 0$

Separação de Variáveis: $V(r, \theta) = R(r) \Theta(\theta)$; logo a E.D.P fica:

$$\underbrace{\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right)}_{\text{const } l(l+1)} + \underbrace{\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right)}_{\text{const } -l(l+1)} = 0$$

$$l = 0, 1, 2, 3, 4, \dots$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r \sin \theta}$$

Eq. p/ R:

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = l(l+1)R \Rightarrow R(r) = Ar^l + \frac{B}{r^{l+1}}$$

↑ infinitas soluções

Eq. p/ Θ :

$$\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = -l(l+1) \sin \theta \cdot \Theta ; \text{ nova variável } x = \cos \theta$$

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} = -\sin \theta \cdot \frac{d}{dx} \Rightarrow -\sin \theta \frac{d}{dx} \left[\sin \theta \left(-\sin \theta \frac{d}{dx} \theta \right) \right] = -l(l+1) \sin \theta \cdot \Theta$$

$$\Rightarrow \frac{d}{dx} \left[\frac{(\sin \theta)^2}{1-x^2} \frac{d\theta}{dx} \right] = -l(l+1) \Theta \Rightarrow \frac{d}{dx} \left[(1-x^2) \frac{d\theta}{dx} \right] = -l(l+1) \Theta$$

→ p/ cada l tem uma nova equação.

Solução: $\Theta(x) = P_l(x)$

↳ Polinômio de Legendre

$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2 - 1) \\ P_3(x) = \frac{1}{2}(5x^3 - 3x) \\ \vdots \end{cases}$$

Formula de Rodrigues:

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2 - 1)^l$$

Ex: $P_4(x) = ?$; Voltando: $\frac{d}{d\theta} \left(\sin\theta \frac{d\theta}{d\theta} \right) = -l(l+1) \sin\theta$

$\theta(\theta) = P_l(\cos\theta)$ ou seja $\left. \begin{matrix} R(r), \theta(\theta) \end{matrix} \right\}$

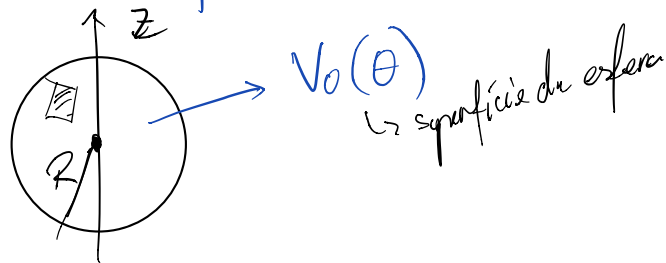
1 ($l=0$)

$\cos\theta$ ($l=1$)

$\frac{1}{2} (3[\cos\theta]^2 - 1)$ ($l=2$)

$V(r, \theta) = R(r) \cdot \theta(\theta)$

$V = \sum_{l=0}^{\infty} \left(\underbrace{A_l r^l}_{\substack{0 \text{ no } 0 \\ \text{div no inf}}} + \underbrace{\frac{B_l}{r^{l+1}}}_{\substack{\text{div } 0 \\ 0 \text{ no inf}}} \right) \cdot P_l(\cos\theta)$ solução geral



Ex1: Potencial em uma cavidade esférica

$V(r, \theta)$ dentro da esfera. $\hookrightarrow$ origem está incluída

$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) \cdot P_l(\cos\theta)$ (geral)

diverge na origem $\rightarrow$

é imposto como condição de contorno que $B_l = 0 \forall l$ $\forall$ que o potencial não diverja na origem.

$\Rightarrow V = \sum_{l=0}^{\infty} A_l r^l \cdot P_l(\cos\theta)$; Condição $V(R, \theta) = V_0(\theta) \Rightarrow V_0(\theta) = \sum_{l=0}^{\infty} \underbrace{A_l R^l}_{\substack{\text{coeficiente} \\ \text{de}}} P_l(\cos\theta)$

$\Rightarrow \sum_{l=0}^{\infty} a_l P_l(\cos\theta) = V_0(\theta) \rightarrow$ Série de Legendre $\nearrow$ intervalo $[-1, 1]$

polinômios são ortogonais entre si

$\int_{-1}^1 P_m(x) P_n(x) dx = \delta_{mn} \cdot \frac{2}{2l+1}$ ortogonalidade

$\hookrightarrow \int_0^\pi P_m(\cos\theta) P_n(\cos\theta) \sin\theta d\theta = \delta_{mn} \cdot \frac{2}{2m+1}$

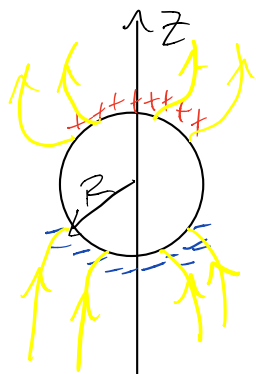
Ex1: mostrar que

$a_l = \frac{2l+1}{2} \int_0^\pi V_0(\theta) P_l(\cos\theta) \sin\theta d\theta$

Ex: $V_0(\theta) = \cos\theta$ $\alpha_1 = 1$
 $\alpha_m = 0$ $\forall m \neq 1$

Aula 26/03:

Esfera condutora em um campo elétrico uniforme:



$$\vec{E} = E_0 \hat{z}$$

Calcular $V(r, \theta)$

$V = 0$ na superfície $(r=R)$

possui simetria em z

$V(r, \theta)$ $\forall r > R$
 $V_{in}(r, \theta)$ $\forall r < R$

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$V_{in}(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \quad r < R \quad B_l = 0, \forall l$$

$$\Rightarrow V_{in}(R, \theta) = \sum_{l=0}^{\infty} \underbrace{A_l R^l}_{\alpha_l} P_l(\cos\theta) = 0$$

$$\alpha_l = 0, \forall l \Rightarrow A_l R^l = 0$$

$\{P_l\}$ base; ortogonal completa

$$\Rightarrow \underline{A_l = 0}$$

$$\boxed{V_{in}(r, \theta) = 0}$$

Fora da esfera: $V = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \quad r > R$

Condição de Contorno:

(a) $V = 0$ $\forall r = R$

(b) $V \rightarrow -E_0 z = -E_0 r \cos\theta$ quando $r \gg R$

$\nearrow$ campo constante

$$V(R, \theta) = 0 = \sum_{l=0}^{\infty} \underbrace{\left(A_l R^l + \frac{B_l}{R^{l+1}} \right)}_{\alpha_l} P_l(\cos \theta) = 0 \quad \left\{ \begin{array}{l} \text{Essa soma só pode ser} \\ \text{zero se } \alpha_l = 0, \forall l \end{array} \right.$$

$$\Rightarrow A_l R^l + \frac{B_l}{R^{l+1}} = 0 \Rightarrow A_l R^l = -\frac{B_l}{R^{l+1}} \Rightarrow \boxed{B_l = -A_l R^{2l+1}}$$

$$\text{Então temos que: } V(r, \theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos \theta)$$

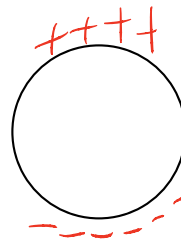
$$r \gg R \rightarrow \text{se resumir a analisar: } \frac{R^{2l+1}}{r^{l+1}} = \left(\frac{R}{r} \right)^{l+1} \cdot R^l \xrightarrow{R/r \rightarrow 0} 0 \text{ p/ todo } l$$

$$V(r, \theta) \rightarrow \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) = A_0 + A_1 r \cos \theta + \dots = -E_0 r \cos \theta$$

$$\Rightarrow A_0 = 0; \underline{A_1 = -E_0} \Rightarrow \boxed{A_{l \neq 1} = 0}$$

$$\text{logo: } \left\{ \begin{array}{l} V(r, \theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta \\ V(r, \theta) = 0 \text{ p/ } r < R \end{array} \right.$$

$$= -E_0 z + \frac{E_0 R^3}{r^2} \cos \theta$$

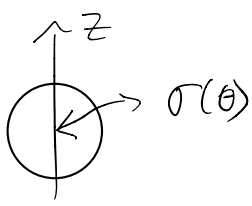


$$\sigma = -\epsilon_0 \vec{\nabla} V \cdot \hat{n} = -\epsilon_0 \frac{\partial V}{\partial r} \Big|_{r=R}$$

$$\stackrel{Ex}{=} 3\epsilon_0 E_0 \cos \theta$$

↳ potencial gerado pela distribuição de carga na esfera devido ao campo elétrico.

Exemplo 2:



$$V_{in}(r, \theta) \quad r \leq R$$

$$V_{out}(r, \theta) \quad r \geq R$$

$$V(r, \theta) = \sum \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

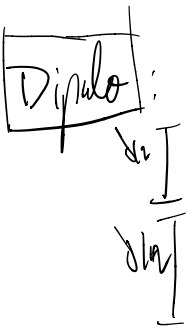
$$V_{in} = \sum (A_\ell r^\ell) P_\ell$$

condição de contorno

$$V_{in}(R, \theta) = V_{out}(R, \theta) \rightarrow \text{continuidade do potencial}$$

$$V_{out} = \sum \left(\frac{B_\ell}{r^{\ell+1}} \right) P_\ell$$

Multipolos:



$$V(\vec{r}) ; V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$\vec{r} = \frac{d}{2} \hat{z} + \vec{r}_\perp \quad \vec{r}_2 = \frac{d}{2} \hat{z} + \vec{r}$$

$$\left\{ \begin{array}{l} \vec{r}_1 = \vec{r} - \frac{d}{2} \hat{z} \\ \vec{r}_2 = \vec{r} + \frac{d}{2} \hat{z} \end{array} \right\} \Rightarrow$$

$$r_1^2 = r^2 - rd \cos \theta + d^2/4 = r^2 \left(1 - \frac{d}{r} \cos \theta + \frac{d^2}{4r^2} \right)$$

$$r_2^2 = r^2 + rd \cos \theta + d^2/4 = r^2 \left(1 + \frac{d}{r} \cos \theta + \frac{d^2}{4r^2} \right)$$

No limite $r \gg d$

$$r_1^2 \approx r^2 \left(1 - \frac{d}{r} \cos \theta \right)$$

$$r_2^2 \approx r^2 \left(1 + \frac{d}{r} \cos \theta \right)$$

$\Rightarrow$

$$\frac{1}{r_1} = \frac{1}{r} \left(1 - \frac{d}{r} \cos \theta \right)^{-1/2} \xrightarrow{\frac{d}{r} \ll 1} \frac{1}{r_1} \approx \frac{1}{r} \left(1 + \frac{1}{2} \frac{d}{r} \cos \theta \right)$$

$$\frac{1}{r_2} \approx \frac{1}{r} \left(1 - \frac{1}{2} \frac{d}{r} \cos \theta \right)$$

$$\frac{1}{r_1} - \frac{1}{r_2} \approx \frac{d}{r^2} \cos \theta$$

$$\Rightarrow V(r, \theta) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q \cdot d \cdot \cos\theta}{r^2}$$

Def: $\vec{P} = q \cdot d \hat{z}$
 $\hat{r} = \frac{\vec{r}}{r}$

$$\Rightarrow V(r, \theta) = \frac{1}{4\pi\epsilon_0} \cdot \frac{\vec{P} \cdot \hat{r}}{r^2}$$

Vale no limite $d \rightarrow 0$
 $q \rightarrow \infty$ $\left\{ \begin{array}{l} dq = \text{const} \end{array} \right.$

monopolo • $V \approx \frac{Q}{r}$;

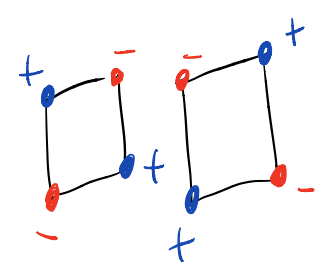
dipolo ±: $V \approx \frac{|\vec{P}|}{r^2}$; $Q_{\text{total}} = 0$

Quadrupolo

$\vec{P} \uparrow \downarrow -\vec{P}$; $Q_{\text{total}} = 0$ $\vec{P}_{\text{total}} = 0$ (2 dipolos opostos)

$d \ll R$
 $V \approx \frac{1}{r^3}$ (4 monopolos)

Octopolo

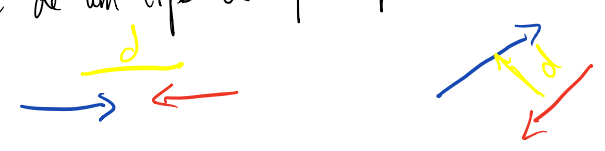


2 quadrupolos opostos
 4 dipolos

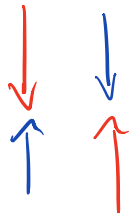
$\Rightarrow$ $V \approx \frac{1}{r^4}$

→ o número de maneiras de montar os polos dependem dos orbitais da mecânica quântica.

Mais de um tipo de quadrupolo

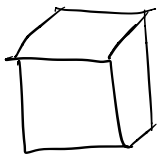


Mais de um tipo de octopolo:



Hexadecapolo: 2 octopolos opostos

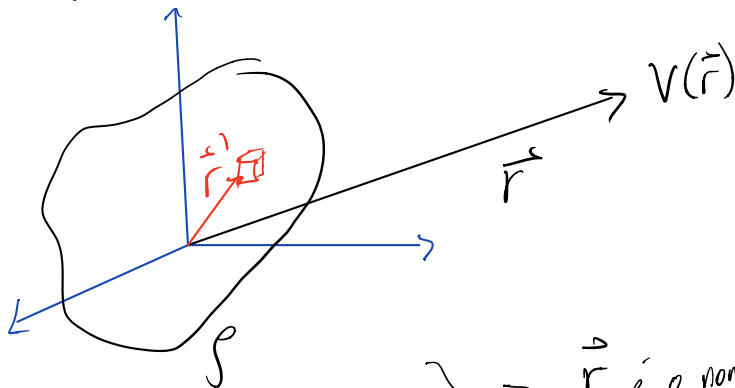
$$V \approx \frac{1}{r^5}$$



4 quadripolos, 8 dipolos, 16 monopolos

Muitas vezes podemos ter um corpo que em pontos suficientemente distantes o potencial se comporta como o potencial de algum multipolo. O exemplo mais concreto disso é o potencial de algumas moléculas.

Em geral



$$V(\vec{r}) = \alpha_1 \frac{1}{r} + \alpha_2 \frac{1}{r^2} + \alpha_3 \frac{1}{r^3} + \dots$$

$\vec{r}$ é o ponto de interesse para calcular o potencial
 $\vec{r}'$ é um vetor que varre todo o corpo e distribuição de carga.

Expansão em multipolos:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3x$$

$$|\vec{r} - \vec{r}'|^2 = r^2 + (r')^2 - 2rr'\cos\theta = r^2 \left[1 + \underbrace{\left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos\theta}_{\epsilon} \right]$$

$\epsilon \rightarrow$ número pequeno

$$\underline{r \gg r'}$$

$$|\vec{r} - \vec{r}'| = r(1 + \epsilon)^{1/2}$$

$$\epsilon = \left(\frac{r'}{r} \right) \left(\frac{r'}{r} - 2\cos\theta \right)$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} (1+\epsilon)^{-1/2} \quad \hookrightarrow \text{expans\~ao em S\~erie de Taylor}$$

$$f(y+\epsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(y)}{n!} \epsilon^n$$

$$(1+\epsilon)^{-1/2} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + O(\epsilon^4)$$

$$\epsilon = \underbrace{\left(\frac{r'}{r}\right)}_a \left(\underbrace{\frac{r'}{r}}_b - \underbrace{2\cos\theta}_b\right)$$

$$(1+\epsilon)^{-1/2} = 1 + a \left[\frac{1}{2}(2b-a) + \frac{3}{8}a(a-2b)^2 + \frac{5}{16}a^2(2b-a)^3 + \dots \right]$$

$$= 1 + \underbrace{a}_P b + \underbrace{a^2}_{P_2(b)} \underbrace{\frac{(3b^2-1)}{2}}_{P_2(b)} + \underbrace{a^3}_{P_3(b)} \underbrace{\frac{(5b^3-3b)}{2}}_{P_3(b)} + \underbrace{a^4}_{P_4(b)} (\dots)$$

polin\~omios de Legendre

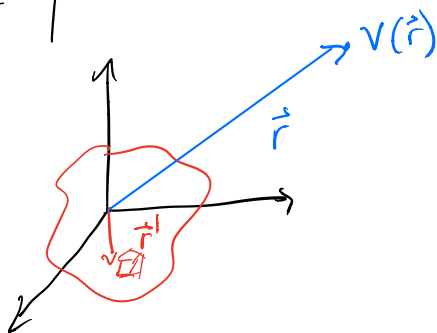
$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta) \rho(\vec{r}') d^3x$$

$$= \frac{1}{4\pi\epsilon_0} \left[\underbrace{\frac{1}{r} \int \rho d^3x}_{Q_{\text{total}}} + \frac{1}{r^2} \int r' \cos\theta \rho(r') d^3x + O\left(\frac{1}{r^3}\right) \right]$$

Aula 2 de Abril:

Vimos que:



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta(\vec{r}, \vec{r}')) \rho(\vec{r}') d\vec{v}'$$

$$= \underbrace{\frac{1}{4\pi\epsilon_0} \frac{1}{r} \int P_0 \rho(\vec{r}') d\vec{v}'}_{\text{potencial de monopolo}} + \underbrace{\frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' P_1(\cos\theta) \rho(\vec{r}') d\vec{v}'}_{\text{dipolo}} + \underbrace{\frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int r'^2 P_2(\cos\theta) \rho(\vec{r}') d\vec{v}'}_{\text{quadrupolo}} + \dots$$

$$V(\vec{r}) = V_{\text{monop}} + V_{\text{dip}} + V_{\text{quad}} + \dots$$

$\frac{1}{r} \quad \frac{1}{r^2} \quad \frac{1}{r^3}$

V_{monop}

$$\int (r')^0 P_0 \rho(\vec{r}') d\vec{v}' = \int \rho(\vec{r}') d\vec{v}' = Q \Rightarrow \boxed{V_{\text{monop}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r} Q}$$

V_{dip}

$$\int (r')^1 P_1(\cos\theta) \rho(\vec{r}') d\vec{v}' = \int r' \cos\theta(\vec{r}, \vec{r}') \rho(\vec{r}') d\vec{v}'; \Rightarrow \vec{r}' \cdot \frac{\vec{r}}{r} = r' \cos\theta$$

$$\Rightarrow \int \vec{r}' \cdot \frac{\vec{r}}{r} \rho(\vec{r}') d\tau' = \frac{\vec{r}}{r} \cdot \underbrace{\int \vec{r}' \rho(\vec{r}') d\tau'}_{\text{vetor que depende de } \rho}$$

Definimos então

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau' \equiv \text{momento de dipolo}$$

$$\Rightarrow \frac{\vec{r}}{r} \cdot \vec{p} \Rightarrow \boxed{V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\vec{p} \cdot \vec{r}}{r}}$$

$$Q = \int \rho d\tau' \quad \vec{p} = \int \vec{r}' \rho d\tau'; \quad \begin{aligned} P_x &= \int x' \rho d\tau' \\ P_y &= \int y' \rho d\tau' \\ P_z &= \int z' \rho d\tau' \end{aligned}$$

Notação: $x = x_1; x_2 = y; x_3 = z$

$$\boxed{P_i = \int x'_i \rho(\vec{r}') d\tau'; i=1,2,3} \rightarrow V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \sum_{i=1}^3 \frac{P_i x_i}{r}$$

Vamos mostrar que

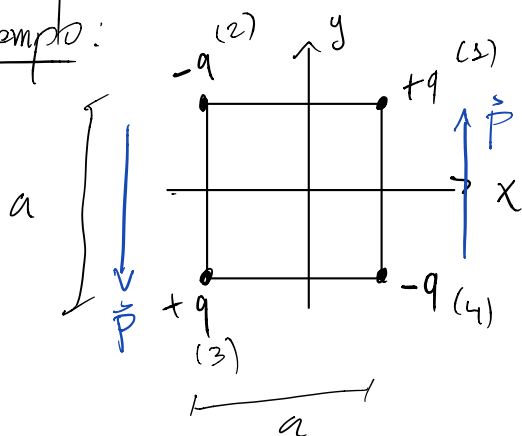
$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r^3} \cdot \frac{1}{2} \sum_i \sum_j Q_{ij} \frac{x_i^2 x_j^2}{r^2}; \text{ onde } \boxed{Q_{ij} = \int (3x'_i x'_j - \delta_{ij} r'^2) \rho(\vec{r}') d\tau'}$$

$\nearrow$ delta de Kronecker

octupolo

$$V = \frac{1}{8} \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{r^4} \left(\frac{1}{2} \sum_{ijk} Q_{ijk} \frac{x_i x_j x_k}{r^3} \right)$$

Exemplo:



$$Q = 0$$

$$\vec{P} = 0$$

$$Q_{ij} = ?$$

$$Q_{ij} = \sum_{\ell=1}^4 \left(3x_i^{(\ell)} x_j^{(\ell)} - r^{(\ell)2} \delta_{ij} \right) q_{\ell}$$

curvas

$$Q_{xx} = \sum_{\ell} \left(3x^{\ell} x^{\ell} - (r^{(\ell)2}) \right) q_{\ell}$$

$$\left\{ \begin{array}{l} x^1 \cdot x^1 = \left(\frac{a}{2}\right)^2 \\ x^2 \cdot x^2 = \left(-\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 \\ x^3 \cdot x^3 = x^4 \cdot x^4 = \left(\frac{a}{2}\right)^2 \\ (r^{(1)})^2 = \left(\frac{\sqrt{2}a}{2}\right)^2 = (r^{(2)})^2 = (r^{(3)})^2 = (r^{(4)})^2 \end{array} \right.$$

$$= 3 \left(\frac{a}{2}\right)^2 - \underbrace{\left(\frac{\sqrt{2}a}{2}\right)^2}_{=6} \sum_{\ell=1}^4 q_{\ell} = 0 \quad \text{Analogamente } Q_{xx} = Q_{yy} = Q_{zz} = 0$$

$$Q_{zx} = Q_{zy} \quad \left| \quad Q_{zx} = \sum_{\ell} \left(3 \cancel{x^{\ell}}^0 x^{\ell} - (r^{\ell})^2 \cancel{\delta_{ij}}^0 \right) = 0 \right.$$

$$Q_{xy} = Q_{yx} \neq 0$$

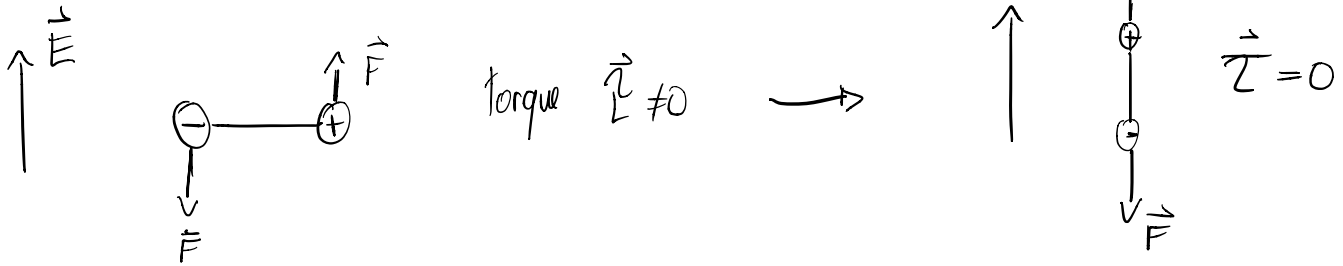
$$Q_{xy} = \sum_{\ell} \left(3x^{\ell} y^{\ell} - \cancel{(r^{\ell})^2 \delta_{12}}^0 \right) q_{\ell} = 3 \left[x^{(1)} y^{(1)} q_1 + x^{(2)} y^{(2)} q_2 + x^{(3)} y^{(3)} q_3 + x^{(4)} y^{(4)} q_4 \right]$$

$$= 3 \left[\left(\frac{1}{2}\right)\left(\frac{a}{2}\right) q + \left(-\frac{a}{2}\right)\left(\frac{a}{2}\right) (-q) + \left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right) q + \left(\frac{a}{2}\right)\left(-\frac{a}{2}\right) (-q) \right] = \underline{3a^2 q = Q_{xy} = Q_{yx}}$$

$$Q_{ij} \rightarrow \begin{pmatrix} 0 & 3a^2q & 0 \\ 3a^2q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Aula 14 de Abril:

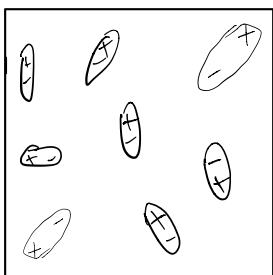
Dieletricos: Dipolo em um campo $\vec{E}$:



$$\vec{P} = q\vec{d}; \text{ momento de dipolo}$$

→ O campo elétrico tende a alinhar o momento de dipolo consigo mesmo
(esse alinhamento pode não ser 100% perfeito)

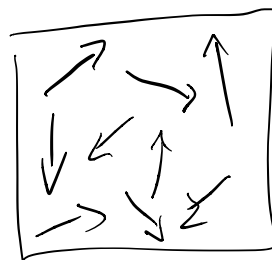
Materiais dielétricos:



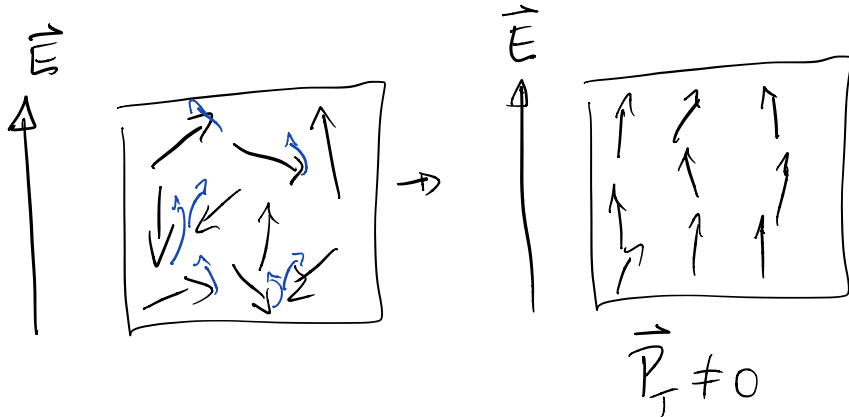
moléculas polares
aleatoriamente distribuídas

Coleção de moléculas polares

$$\vec{P}_T = 0$$



$$\vec{P}_T = \sum_i \vec{P}_i$$



tendência de alinhamento

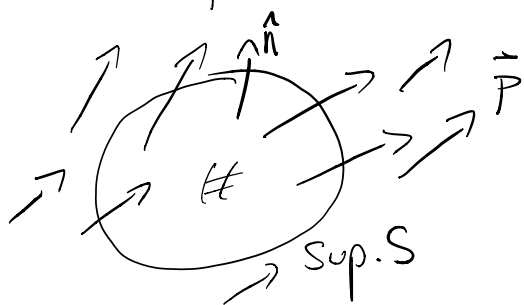
Vetor Polarização (Densidade de Polarização)

$$\vec{P} := \frac{\vec{P}_T \text{ no volume } \Delta V}{\Delta V}$$

$\leadsto$ Campo vetorial genuíno

$\vec{P}$ uniforme $\rightarrow$ densidade de carga apenas na superfície (σ)

$\vec{P}$ não uniforme $\rightarrow$ densidade de carga na superfície e no interior (ρ)

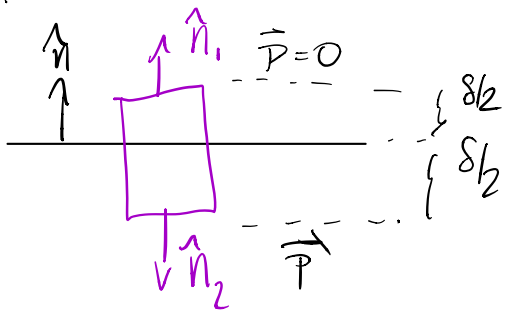
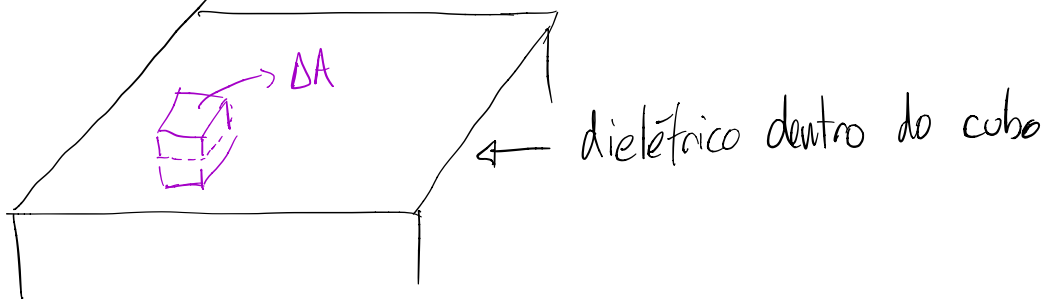


podemos mostrar que

$$\oint_S \vec{P} \cdot \hat{n} ds = -Q_p \text{ (dentro de } S)$$

$Q_p \rightarrow$ carga de polarização

$$\hookrightarrow \oint \vec{P} \cdot \hat{n} ds = \int_{Vol} \vec{\nabla} \cdot \vec{P} d\tau = -Q_p = - \int_{Vol} \rho_p d\tau \Rightarrow \boxed{\vec{\nabla} \cdot \vec{P} = -\rho_p}$$



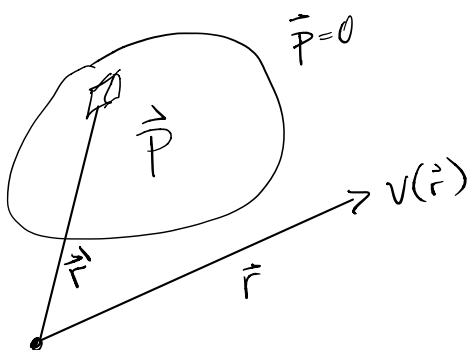
$$\oint \vec{P} \cdot \hat{n} ds = -Q$$

$$\lim_{\delta \rightarrow 0}$$

$$\oint \vec{P} \cdot \hat{n} ds = \int \vec{P} \cdot \hat{n}_1 ds + \int \vec{P} \cdot \hat{n}_2 ds + \int_{\text{latente}} \vec{P} \cdot \hat{n} ds = \vec{P} \cdot \hat{n}_1 \Delta A + \vec{P} \cdot \hat{n}_2 \Delta A = -\vec{P} \cdot \hat{n} \Delta A$$

$\delta \rightarrow 0$

$$Q_A = \sigma_P \cdot \Delta A \rightarrow \boxed{\vec{P} \cdot \hat{n} = \sigma_P}$$



Um volume Δv contribui com $\vec{P} = \vec{P} \Delta v$

$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{|\vec{r}-\vec{r}'|^2} \frac{(\vec{r}-\vec{r}') \cdot \vec{P} \Delta v}{|\vec{r}-\vec{r}'|}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \cdot \vec{P}(\vec{r}') dv'$$

identidades: $\vec{\nabla}' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad (1)$

$$\vec{\nabla} (f \vec{A}) = \vec{\nabla} f \cdot \vec{A} + f \vec{\nabla} \cdot \vec{A} \quad (2)$$

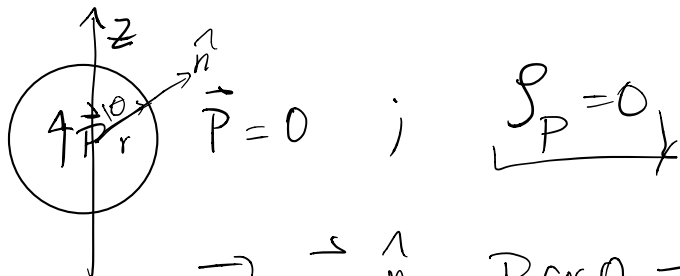
$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \vec{\nabla}' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \cdot \vec{P}(\vec{r}') d\vec{r}'$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\text{Vol}} \vec{\nabla}' \cdot \left(\frac{\vec{P}}{|\vec{r} - \vec{r}'|} \right) d\vec{r}' - \int_{\text{Vol}} \frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla}' \cdot \vec{P} d\vec{r}' \right]$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\oint_{\text{sup}} \frac{\vec{P} \cdot \hat{n}}{|\vec{r} - \vec{r}'|} ds + \int_{\text{Vol}} \frac{1}{|\vec{r} - \vec{r}'|} (-\vec{\nabla}' \cdot \vec{P}) d\vec{r}' \right]$$

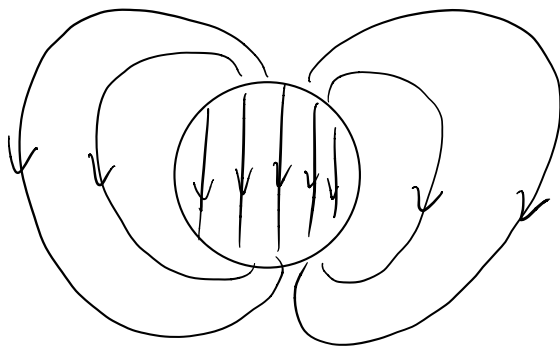
$$\sigma_P = \vec{P} \cdot \hat{n} \quad ; \quad \rho_P = -\vec{\nabla} \cdot \vec{P}$$

Exemplo: Esfera raio R , $\vec{P} = P \hat{z}$, uniforme



$$\rightarrow \vec{P} \cdot \hat{n} = P \cos \theta = \sigma_P$$

$$V(\vec{r}) = \begin{cases} \frac{P}{3\epsilon_0} r \cos \theta & r < R \rightarrow \vec{E} = -\frac{P}{3\epsilon_0} \hat{k} \\ \frac{P}{3\epsilon_0} \frac{R^3}{r^2} \cos \theta & r > R \rightarrow \vec{E} \text{ de dipolo} \end{cases} \quad \vec{P} = \frac{4}{3} \pi R^3 P \hat{k}$$



Aula 16 de Abril:

$$\mathcal{J} = \mathcal{J}_e + \mathcal{J}_p$$

densidade de
carga livre

depende do efeito
da polarização

$$\epsilon_0 \cdot \vec{\nabla} \cdot \vec{E} = \mathcal{J} = \mathcal{J}_e + \mathcal{J}_p = \mathcal{J}_e - \vec{\nabla} \cdot \vec{P} \Rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \mathcal{J}_e$$

$$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \mathcal{J}_e$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

vetor de
deslocamento

$\Rightarrow$

$$\vec{\nabla} \cdot \vec{D} = \mathcal{J}_e$$

~ escrever as leis de Maxwell em termos das
cargas livres da qual temos controle

$$\vec{P} = \chi_E \epsilon_0 \vec{E} \rightarrow \text{Dielétricos Lineares}$$

$\hookrightarrow$ Sus Elétrica

$\hookrightarrow$ relação linear entre polarização e campo elétrico

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \chi_E \epsilon_0 \vec{E} \rightarrow$$

$$\vec{D} = \epsilon_0 (1 + \chi_E) \vec{E}$$

χ_E depende
do material

podemos definir ainda $\epsilon = \epsilon_0 (1 + \chi_E)$

$$\Rightarrow \vec{D} = \epsilon \vec{E}$$

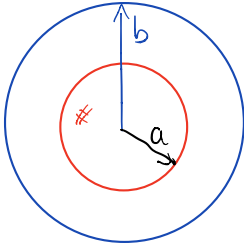
$$K = \frac{\epsilon}{\epsilon_0}$$

→ Delimita o quão diferente o material em questão é do vácuo

→ Constante Dielétrica

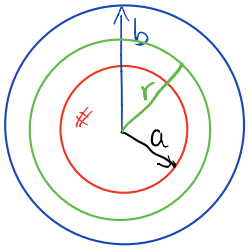
$$K = 1 + \chi_E$$

Exemplo:



Carga Q na esfera $\vec{E} = ?$

$$\vec{\nabla} \cdot \vec{D} = \rho_e \rightarrow \oint_S \vec{D} \cdot d\vec{A} = Q_e \text{ (dentro de } S)$$



$$\oint \vec{D} \cdot \hat{n} ds = D \cdot 4\pi r^2 = Q \Rightarrow \vec{D} = \frac{Q}{4\pi} \cdot \frac{1}{r^2} \hat{r}$$

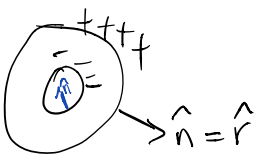
$$\vec{D} = \epsilon \vec{E} \rightarrow \vec{E} = \frac{Q}{4\pi\epsilon} \cdot \frac{1}{r^2} \hat{r} ; \underline{a < r < b}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{r^2} \hat{r} \quad (r > b) \quad \left\{ \begin{array}{l} \vec{P} = \epsilon_0 \chi_E \vec{E} \\ \vec{P} = \frac{\chi_E \epsilon_0 Q}{\epsilon 4\pi} \cdot \frac{1}{r^2} \hat{r} \end{array} \right.$$

↪ não existe polarização do lado de fora.

$$\vec{\nabla} \cdot \vec{P} = -\rho_p \rightarrow \vec{\nabla} \cdot \vec{P} = 0 ; P \sim \frac{1}{r^2} \rightarrow \boxed{\rho_p = 0}$$

$$\vec{P} \cdot \hat{n} = \sigma_p$$



$$\sigma_p \left\{ \begin{array}{l} \frac{\chi_E \epsilon_0 Q}{4\pi\epsilon} \cdot \frac{1}{b^2} \quad r=b \\ -\frac{\chi_E \epsilon_0 Q}{4\pi\epsilon} \cdot \frac{1}{a^2} \quad r=a \end{array} \right.$$

Eletrostática:

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho$$

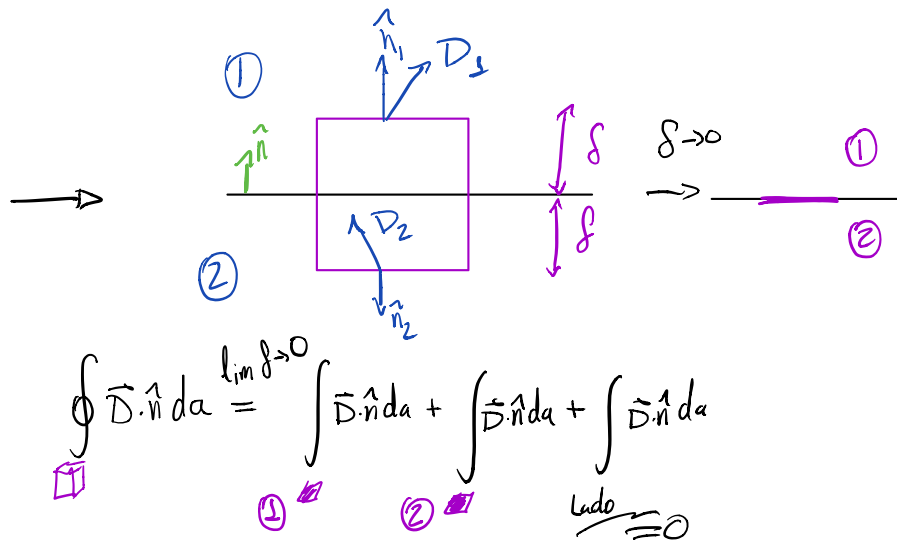
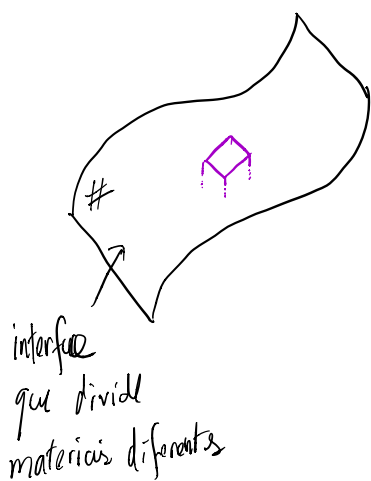
$$\vec{\nabla} \times \vec{E} = 0$$

no vácuo

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{D} = \rho_e \\ \vec{\nabla} \times \vec{E} = 0 \\ \vec{D} = \epsilon \vec{E} \end{array} \right. \quad \text{dielétrico}$$

↑
inf do material

Condições de Contorno:



$$= \vec{D}_1 \cdot \hat{n} \Delta A + \vec{D}_2 \cdot (-\hat{n}) \Delta A = \sigma_e \Delta A$$

$$\Rightarrow \left. \begin{array}{l} D_1^\perp - D_2^\perp = \sigma_e \\ E_1^\parallel - E_2^\parallel = 0 \end{array} \right\} \text{Condições de Contorno}$$

$$\vec{\nabla}^2 V = \frac{1}{\epsilon_0} (\rho_e + \rho_p) ; \text{ supondo que } \rho_e = 0$$

$$\vec{\nabla}^2 V = \frac{1}{\epsilon_0} \rho_p \rightarrow \vec{\nabla}^2 V = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot \vec{P} = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot (\epsilon_0 \chi_e \vec{E})$$

dentro do material dielétrico

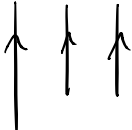
$$\vec{\nabla}^2 V = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot (\epsilon_0 \chi_E \vec{E} \cdot \frac{\vec{e}}{e}) = -\frac{\epsilon_0 \chi_E}{\epsilon_0 \cdot e} \vec{\nabla} \cdot \vec{D} = -\frac{\chi_E}{e} \vec{\nabla} \cdot \vec{D}$$

$$\Rightarrow \vec{\nabla}^2 V = -\frac{\chi_E}{e} \rho_e \rightarrow \boxed{\vec{\nabla}^2 V = 0} \rightarrow \text{com novas condições de contorno}$$

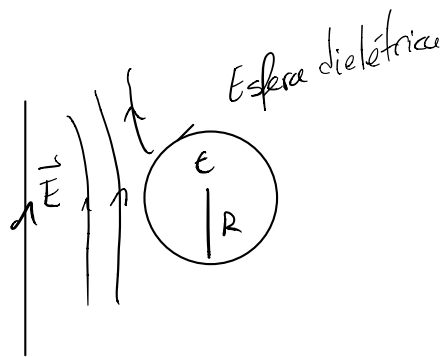
$$\begin{cases} V_1 = V_2 \\ \epsilon_1 \frac{\partial V}{\partial \eta} - \epsilon_2 \frac{\partial V}{\partial \eta} = -\sigma_e \end{cases} \rightarrow \text{na interface}$$

Aula 23 de Abril:

Exemplo:



$$\vec{E} = E_0 \hat{z}$$



$$V_{out} = ?$$

$$V_{in} = ?$$

$$\vec{\nabla}^2 V = 0; \quad V_{in}(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{\tilde{B}_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$V_{out}(r, \theta) = \sum_{l=0}^{\infty} \left(\tilde{A}_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

aplicar condições de contorno para determinar os coeficientes

$$V_{in} \text{ regular em } r=0 \therefore \underline{\tilde{B}_l = 0}$$

$$V_{out} \stackrel{r \gg R}{\simeq} \tilde{A}_0 + \tilde{A}_1 r \cos \theta + \tilde{A}_2 r^2 P_2 + \dots \simeq -E_0 z = -E_0 r \cos \theta \rightarrow \text{campo uniforme}$$

$$\underline{\tilde{A}_0 = 0; \quad \tilde{A}_1 = -E_0; \quad \tilde{A}_n = 0, \forall n \neq 1}$$

$$\Rightarrow V_{out} = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\bullet V_{in}(R, \theta) = V_{out}(R, \theta)$$

$$\sum_l A_l R^l P_l = \underbrace{-E_0 R \cos\theta}_{-E_0 R P_1} + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l$$

$$\left\{ \begin{array}{l} A_l R = -E_0 R + \frac{B_1}{R^2} \quad ; \quad l=1 \\ A_n R^n = \frac{B_n}{R^{n+1}} \quad ; \quad n \neq 1 \end{array} \right\} \quad (1) ; \quad \epsilon \frac{\partial V_{in}}{\partial r} - \epsilon_0 \frac{\partial V_{out}}{\partial r} = 0$$

$$\Rightarrow \epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r} \quad ; \quad \underline{r=R}$$

$$\Rightarrow \underbrace{\frac{\epsilon}{\epsilon_0}}_{K = \text{const. dielétrica}} \frac{\partial V_{in}}{\partial r} = \frac{\partial V_{out}}{\partial r} \quad ; \quad r=R$$

$$\left\{ \begin{array}{l} K A_1 = -E_0 - \frac{2B_1}{R^3} \quad ; \quad l=1 \\ l K A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} \quad ; \quad l \neq 1 \end{array} \right\} \quad (2) \quad \text{Resolvendo as equações}$$

$$\left\{ \begin{array}{l} A_n R^n = \frac{B_n}{R^{n+1}} \Rightarrow A_n R^{2n+1} = B_n \\ l K A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}} \Rightarrow n K \underbrace{A_n R^{2n+1}}_{B_n} = -(n+1) B_n \Rightarrow (K n + n + 1) B_n = 0 \end{array} \right.$$

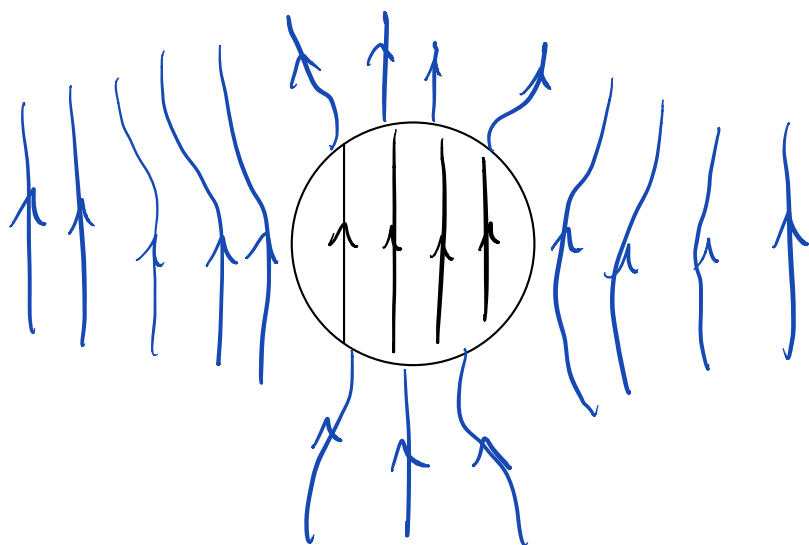
$$\Rightarrow \underline{B_n = 0} \quad ; \quad n \neq 1$$

$$\Rightarrow \underline{A_n = 0} \quad ; \quad n \neq 1$$

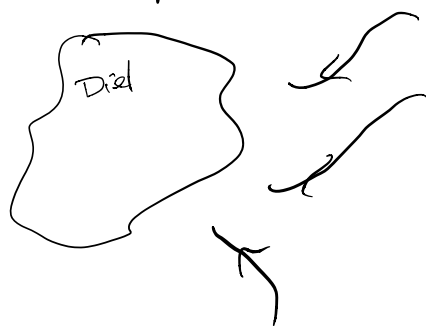
$$\begin{cases} A_1 R - \frac{B_1}{R^2} = -E_0 R \\ KA_1 + 2\frac{B_1}{R^3} = -E_0 \end{cases} \Rightarrow \begin{aligned} A_1 &= \frac{-3}{K+2} E_0 \\ B_1 &= \left(\frac{K-1}{K+2} \right) R^3 E_0 \end{aligned}$$

$$V_{in}(r, \theta) = A_1 r \cos \theta = \frac{-3E_0 z}{K+2} ; \quad \vec{E}_{in} = \frac{3E_0}{K+2} \hat{k} \quad \swarrow \text{campo uniforme}$$

$$V_{out}(r, \theta) = -E_0 z + \frac{B_1}{r^2} \cos \theta = -E_0 z + \left(\frac{K-1}{K+2} \right) R^3 E_0 \cdot \frac{\cos \theta}{r^2} \quad \swarrow \text{campo de dipolo}$$



Energia do Campo Elétrico:



$$\rho_e \rightarrow \hat{\rho}_e = \rho_e + \delta \rho_e$$

$$\int W = \int_{\mathbb{R}^3} \delta \rho_e v dv$$

$$\vec{\nabla} \cdot \vec{D} = \rho ; \quad \tilde{\rho} = \rho + \delta \rho$$

$$\vec{\nabla} \cdot \vec{D} = \hat{\rho} = \underline{\vec{\nabla} \cdot \vec{D} + \delta \rho}$$

$$\vec{\tilde{D}} = \vec{D} + \delta \vec{D} \rightarrow \vec{\tilde{V}} \cdot \vec{\tilde{D}} = \vec{\tilde{V}} \cdot \vec{D} + \vec{\tilde{V}} \cdot \delta \vec{D} \Leftrightarrow \tilde{J} = J + \underline{\vec{\tilde{V}} \cdot \delta \vec{D}}$$

$$\Rightarrow \boxed{\vec{\tilde{V}} \cdot \delta \vec{D} = \delta J}$$

$$\delta W = \int_{\mathbb{R}^3} \vec{\tilde{V}} \cdot \delta \vec{D} \, v \, d\sigma; \quad \vec{\tilde{V}} \cdot (\delta \vec{D} \, v) = (\vec{\tilde{V}} \cdot \delta \vec{D}) v + \delta \vec{D} \cdot \vec{\tilde{V}} v$$

$$\vec{\tilde{V}} \delta \vec{D} \cdot v = \vec{\tilde{V}} (\delta \vec{D} \, v) - \delta \vec{D} \cdot \vec{\tilde{V}} v$$

$$\Rightarrow \delta W = \int_{\mathbb{R}^3} \vec{\tilde{V}} \cdot (\delta \vec{D} \, v) \, d\sigma + \int \delta \vec{D} \cdot \vec{E} \, d\sigma$$

$$= \oint \cancel{\delta \vec{D} \cdot \vec{n} \, d\sigma} + \int \delta \vec{D} \cdot \vec{E} \, d\sigma \Rightarrow \delta W = \int_{\mathbb{R}^3} \delta \vec{D} \cdot \vec{E} \, d\sigma$$

Corrente Elétrica:

1D

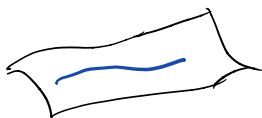
$$\vec{I} = \lambda \vec{v}$$



Corrente que passa por um ponto

2D

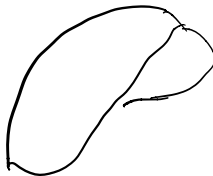
$$\vec{K} = \sigma \vec{v}$$



Corrente que passa por uma curva

3D

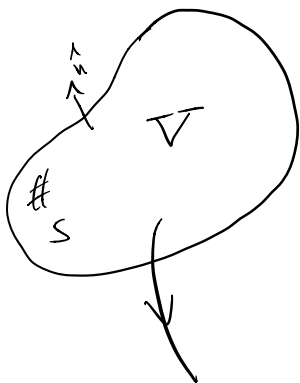
$$\vec{J} = \rho \vec{v}$$



Corrente que passa por uma Superfície

$$I = \int_{\gamma} \vec{K} \cdot d\vec{\ell} \quad ; \quad I = \iint_S \vec{J} \cdot d\vec{a}$$

Conservação da Carga



$Q[V] =$ carga no volume V

A carga neste corpo só pode variar se entrar carga do lado de fora pr o lado de dentro ou se sair carga do lado de dentro pr lado de fora.

Se a carga varia sem esse transporte, acontece na criação/destruição de cargas, o que é impossível logo a carga de um corpo só pode variar mediante a uma corrente na superfície

$$\frac{dQ[V]}{dt} = -I[S] \quad \text{na definição (dar nome a algo)}$$

$$\Rightarrow \frac{d}{dt} \int_V \rho d\sigma = - \oint_S \vec{J} \cdot d\vec{a} = - \int_V \vec{\nabla} \cdot \vec{J} d\sigma \Rightarrow \int_V \frac{\partial \rho}{\partial t} d\sigma = - \int_V \vec{\nabla} \cdot \vec{J} d\sigma$$

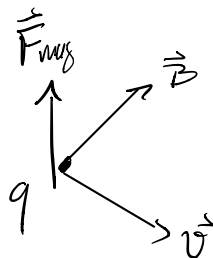
$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) d\sigma = 0 ; \forall \text{ volume } V \Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \Rightarrow$$

$$\boxed{\vec{\nabla} \cdot \vec{J} = - \frac{\partial \rho}{\partial t}}$$

Equação da Continuidade

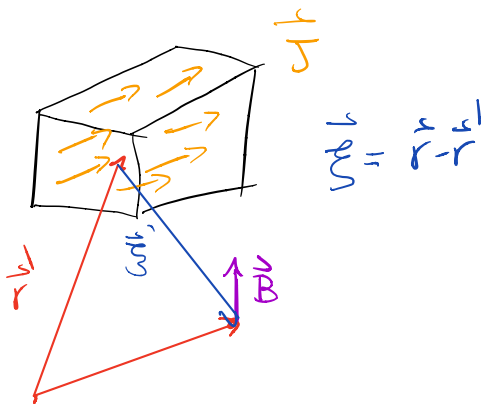
Força Magnética

$$\boxed{\vec{F}_{\text{mag}} = q \vec{v} \times \vec{B}}$$



Fonte do campo magnético: Correntes $\vec{J}$, $\vec{K}$, I

Lei de Biot-Savart:



$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\sigma}$$



$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} da}$$



$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\gamma} \frac{\vec{I}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dl}$$

Sabemos que: $\vec{\nabla} \cdot \vec{B} = 0$; $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (magnetostática) $\rightarrow$ não há dependência temporal

$$\vec{\nabla}_r \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{\nabla}_r \left(\vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) d\tau'$$

$$\Rightarrow \vec{\nabla}_r \cdot \left(\vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = \cancel{(\vec{\nabla}_r \times \vec{J}) \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}} - \vec{J} \cdot \vec{\nabla}_r \times \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right)$$

$$\vec{\nabla}_r \times \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \frac{\mu_0}{4\pi} \int \vec{\nabla}_r \times \left(\vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) d\tau' = \\ &= \vec{J} \underbrace{\vec{\nabla}_r \cdot \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right)}_{=0} - (\vec{J} \cdot \vec{\nabla}_r) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \end{aligned}$$

$$\vec{\nabla}_r \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \begin{cases} \infty, & r = r' \\ 0, & r \neq r' \end{cases} = \underline{4\pi \delta(\vec{r} - \vec{r}')} \quad \cancel{\underline{\quad}}$$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J} \cdot 4\pi \delta(\vec{r} - \vec{r}') d\tau' - \underbrace{\int (\vec{J} \cdot \vec{\nabla}_r) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}}_{=0}$$

$$\rightarrow \mu_0 \vec{J} = \vec{\nabla} \times \vec{B}$$

Mostraremos que:

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla} \times \vec{A}} \text{ para algum } \vec{A} \rightarrow \text{Potencial Vetorial}$$

$$\vec{B} = \text{dado}; \vec{A}_1; \vec{\nabla} \times \vec{A}_1 = \vec{B} \checkmark$$

$$\vec{A}_2 = \vec{A}_1 + \vec{\nabla} f; f = \text{qualquer}; \vec{\nabla} \times \vec{A}_2 = \vec{B}?$$

$\vec{A}$ não é único

$$\vec{\nabla} \times \vec{A}_2 = \vec{\nabla} \times (\vec{A}_1 + \vec{\nabla} f) = \vec{\nabla} \times \vec{A}_1 + \underbrace{\vec{\nabla} \times \vec{\nabla} f}_{=0} = \vec{\nabla} \times \vec{A}_1 = \vec{B} \checkmark$$

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0 \checkmark$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

Suponha que $\vec{\nabla}_0 \vec{A}_0 \neq 0$; sena que $\exists \lambda(x,y,z)$ tal que $\vec{A} = \vec{A}_0 + \vec{\nabla} \lambda; \vec{\nabla} \cdot \vec{A} = 0?$
 $\hookrightarrow \lambda = \lambda(r)$

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\vec{\nabla} \cdot (\vec{A}_0 + \vec{\nabla} \lambda) = 0 \rightarrow \text{achar um } \lambda \text{ tal que isso seja verdade}$$

$$\vec{\nabla}_0 \vec{A}_0 + \vec{\nabla} \cdot \vec{\nabla} \lambda = 0 \Rightarrow \underbrace{\lambda(r) + \nabla^2 \lambda = 0}_{\text{Dado}} \Rightarrow \boxed{\vec{\nabla}^2 \lambda = -\alpha} \Rightarrow \lambda = \frac{1}{4\pi} \int_V \frac{\alpha(r')}{|\vec{r} - \vec{r}'|} d\omega'$$

Eq. de Poisson $\vec{\nabla} \cdot \vec{A}_0$

$$\Rightarrow \boxed{\vec{A} = \vec{A}_0 + \vec{\nabla} \lambda; \vec{\nabla} \cdot \vec{A} = 0} \leadsto \text{construção de um novo } \vec{A} \text{ que satisfaz } \vec{\nabla} \cdot \vec{A} = 0$$

Voltando temos:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

$$\vec{\tilde{A}} = \vec{A} + \vec{\nabla} \lambda$$

Sempre existe um $\vec{A}$ tq $\vec{\nabla} \cdot \vec{A} = 0$

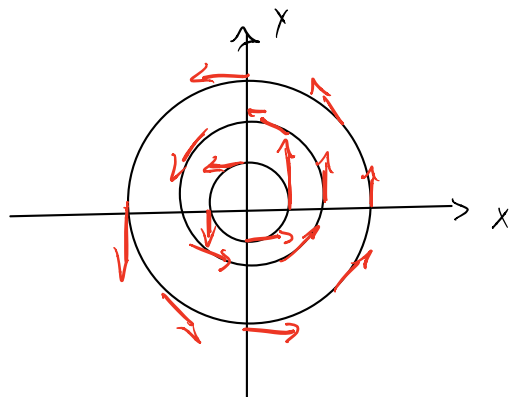
$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}} \quad \text{as 3 eq de Laplace}$$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'}$$

Se fizermos $\vec{\nabla} \times \vec{A}$ caímos na Lei de Biot-Savart

Exemplo: Coord (r, ϕ, z) ; $r = \sqrt{x^2 + y^2}$

$$\vec{J} = \frac{\alpha}{\mu_0} \cdot \frac{1}{r^2} \hat{\phi}$$



Encontre $\vec{B}$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$

$$\begin{cases} \vec{\nabla} \times \vec{B} = \frac{\alpha}{r^2} \hat{\phi} \\ \vec{\nabla} \cdot \vec{B} = 0 \end{cases}$$

Simetrias

- Translação em z
- Rotação em z

$$\vec{B} = B_r \hat{r} + B_\phi \hat{\phi} + B_z \hat{z}$$

$$\Rightarrow B_r = B_r(r)$$

$$B_\phi = B_\phi(r)$$

$$B_z = B_z(r)$$

~ hipóteses

$$(\vec{\nabla} \times \vec{B})_r = \frac{1}{r} \frac{\partial}{\partial \phi} B_z - \frac{\partial B_\phi}{\partial z} = \mu_0 J_r = 0 \quad (\perp)$$

$$(\vec{\nabla} \times \vec{B})_\varphi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \mu_0 J_\varphi = \frac{\alpha}{r^2} \quad (2)$$

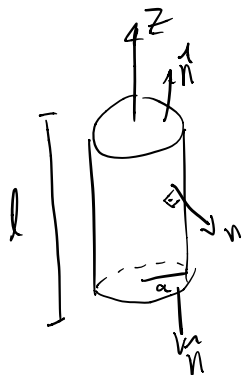
$$(\vec{\nabla} \times \vec{B})_z = \frac{1}{r} \left[\frac{\partial}{\partial r} (r B_\varphi) - \frac{\partial B_r}{\partial \varphi} \right] = \mu_0 J_z = 0 \quad (3)$$

$$-\frac{dB_z}{dr} = \alpha \frac{1}{r^2} \Rightarrow B_z = \frac{\alpha}{r} + C_1 \quad \begin{array}{l} B \rightarrow 0 \text{ quando } r \rightarrow \infty \\ \Rightarrow C_1 = 0 \end{array} \Rightarrow \boxed{B_z = \frac{\alpha}{r}}$$

$$\Rightarrow \boxed{B_\varphi = \frac{C_2}{r}} \quad \left| \begin{array}{l} \vec{\nabla} \cdot \vec{B} = 0 \\ = \frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{1}{r} \frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} = 0 \\ \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r B_r) = 0 \Rightarrow \boxed{B_r = \frac{C_3}{r}} \end{array} \right.$$

$$\vec{B} = (B_r, B_\varphi, B_z) = \left(\frac{C_3}{r}, \frac{C_2}{r}, \frac{\alpha}{r} \right); \quad \begin{array}{l} C_2 = ? \\ C_3 = ? \end{array} \quad \text{em } \mathbb{R}^3 - \{0\} \quad \left\{ \begin{array}{l} \text{oq. esta acontecendo na} \\ \text{origem?} \\ \hookrightarrow \text{analisar forma integral} \end{array} \right.$$

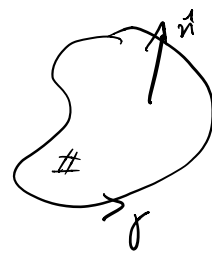
$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \int_V \vec{\nabla} \cdot \vec{B} = \oint_S \vec{B} \cdot \vec{n} da = 0 \quad \forall S$$



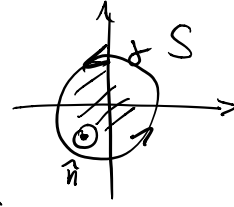
$$\oint \vec{B} \cdot \vec{n} da = \int_{\text{base}} + \int_T + \int_B = \int_{\text{base}} \frac{C_3}{a} \vec{r} \cdot \vec{n} + \overbrace{\int_T \frac{\alpha}{r} \vec{z} \cdot \vec{z} + \int_B \frac{\alpha}{r} \vec{z} \cdot (-\vec{z})}^{0, \text{ se cancelarem}} = \frac{C_3 2\pi a l}{a}$$

$$\Rightarrow \oint \vec{B} \cdot \vec{n} ds = 2\pi C_3 l \Rightarrow \text{isso é nulo apenas se } \boxed{C_3 = 0}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \frac{\alpha}{r^2} \hat{\phi} \rightarrow \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot \hat{n} da$$



$$\oint_{\gamma} \vec{B} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} \Rightarrow \oint_{\gamma} \vec{B} \cdot d\vec{\ell} = \mu_0 I$$



$$\vec{\ell} = \hat{\phi} d\ell \quad \oint \vec{B} \cdot \hat{\phi} d\ell = \oint B_{\phi} d\ell$$

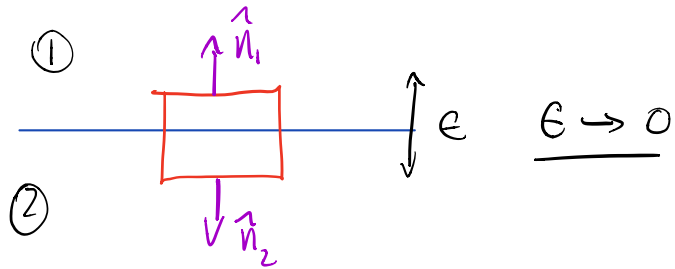
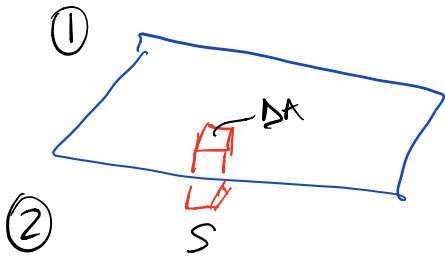
$$= \oint \frac{C_2}{a} d\ell = \frac{C_2}{a} \cdot 2\pi a = 2\pi C_2$$

$$\oint_{\gamma} \vec{B} \cdot d\vec{\ell} = 2\pi C_2 \equiv 0 \rightarrow \boxed{C_2 = 0}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\alpha}{r} \hat{z}}$$

Condições de Contorno p/ $\vec{B}$ e $\vec{A}$:

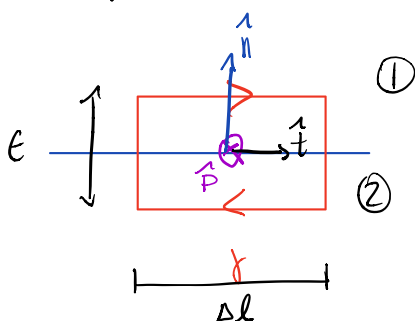
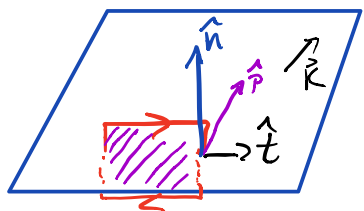
$$\vec{\nabla} \cdot \vec{B} = 0 ; \quad \oint_S \vec{B} \cdot d\vec{a} = 0$$



$$\oint_S \vec{B} \cdot d\vec{a} = \int_T \vec{B}_1 \cdot \hat{n} da + \int_B \vec{B}_2 \cdot \hat{n} da + \int_{lateral} \vec{B} \cdot \hat{n} da \xrightarrow{\epsilon \rightarrow 0} (\vec{B}_1 - \vec{B}_2) \cdot \hat{n} \Delta A = 0$$

$$\Rightarrow \boxed{\vec{B}_1^{\perp} = \vec{B}_2^{\perp}}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} ; \quad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$$



$$\oint \vec{B} \cdot d\vec{\ell} = \int \vec{B}_1 \cdot \hat{t}_1 d\ell + \int \vec{B}_2 \cdot \hat{t}_2 d\ell + \int \vec{B} \cdot d\vec{\ell} \xrightarrow{\epsilon \rightarrow 0} \mu_0 I$$

$$= (\vec{B}_1 - \vec{B}_2) \cdot \hat{t} \Delta l = \mu_0 \vec{K} \cdot \vec{P} \Delta l$$

$$\boxed{(\vec{B}_1 - \vec{B}_2) \cdot \hat{t} = \mu_0 \vec{K} \cdot \vec{P}}$$

$\hat{t} \equiv$ qualquer vetor tangente

$$\vec{P} = \hat{n} \times \hat{t} ; \quad \vec{K} \cdot \vec{P} = \vec{K} \cdot (\hat{n} \times \hat{t}) = \hat{t} \cdot (\vec{K} \times \hat{n})$$

$$\Rightarrow \begin{cases} (\vec{B}_1 - \vec{B}_2) \cdot \hat{t} = \mu_0 (\vec{K} \times \hat{n}) \cdot \hat{t} \\ (\vec{B}_1 - \vec{B}_2) \cdot \hat{n} = \mu_0 \underbrace{(\vec{K} \times \hat{n}) \cdot \hat{n}}_{=0} \end{cases} \Rightarrow \boxed{\vec{B}_1 - \vec{B}_2 = \mu_0 \vec{K} \times \hat{n}}$$

Condições de Contorno para $\vec{A}$

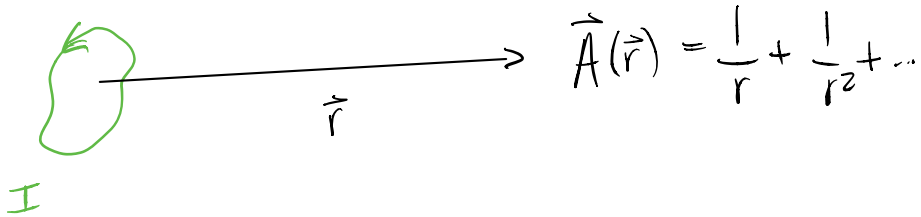
$$\vec{\nabla} \cdot \vec{A} = 0 \rightarrow \boxed{A_1^\perp = A_2^\perp} \quad \text{p/ os pontos da interface}$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \Rightarrow \oint \vec{A} \cdot d\vec{\ell} = \int \vec{B} \cdot \vec{n} ds \rightarrow \boxed{A_1'' = A_2''}$$

$$\Rightarrow \boxed{\vec{A}_1 = \vec{A}_2}$$

Potencial Vetor é contínuo ao longo da superfície

Aproximação de multipolos:



↗ Analisar um loop de corrente de uma distância considerável e analisar qual o potencial vetor gerado.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{\ell}'}{|\vec{r} - \vec{r}'|} \quad \text{a aproximação entra em } |\vec{r} - \vec{r}'|;$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

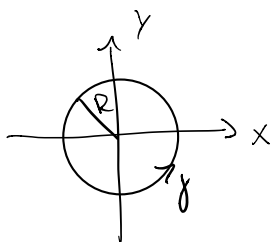
$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\vec{\ell}' + \frac{1}{r^2} \oint r' \cos\theta' d\vec{\ell}' + \dots \right]$$

$$d\vec{\ell} = (dx, dy, dz) \xrightarrow{\oint} (x_f - x_i, y_f - y_i, z_f - z_i) = \vec{0} \quad (\text{curva fechada})$$

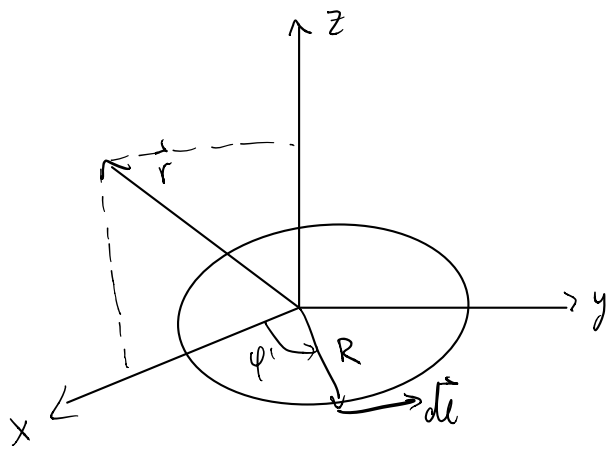
$$\vec{A}_{\text{dip}} = \frac{\mu_0 I}{4\pi} \cdot \frac{1}{r^2} \oint r' \cos\theta' d\vec{\ell}'$$

$$\oint (\vec{r}' \cdot \hat{r}) d\vec{\ell}' = \vec{F}(\hat{r})$$

Exemplo:



Escolher (x, y, z) tal que $\vec{r} = (x, 0, z)$



$$d\vec{l}' = R d\phi \hat{\phi}$$

$$= R d\phi (-\sin\phi \hat{x} + \cos\phi \hat{y})$$

$$\vec{r}' = R(\cos\phi \hat{x} + \sin\phi \hat{y})$$

$$\hat{r} = \frac{1}{r} (x\hat{x} + z\hat{z})$$

$$\oint (\hat{r} \cdot \vec{r}') d\vec{l}' = -\frac{xR^2}{r} \underbrace{\int_0^{2\pi} \sin\phi \cos\phi d\phi}_{=0} \hat{x} + \frac{xR^2}{r} \underbrace{\int_0^{2\pi} \cos^2\phi d\phi}_{=\pi} \hat{y}$$

$$= \frac{x}{r} \pi R^2 \hat{y}$$

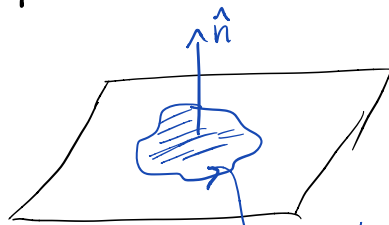
$$= \vec{a} \times \vec{r} = \frac{\vec{a} \times \vec{r}}{r}$$

$$\hat{r} = \frac{1}{r} (x\hat{x} + z\hat{z})$$

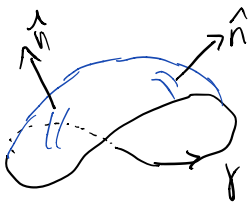
inventor

$$\vec{a} = \pi R^2 \hat{z} = \text{Area} \cdot \hat{z}$$

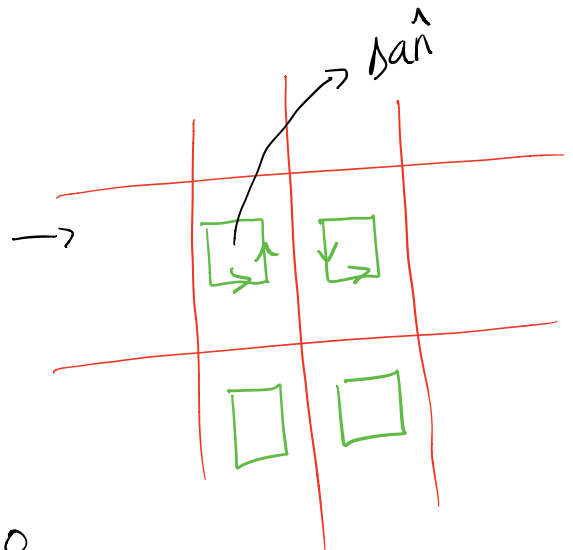
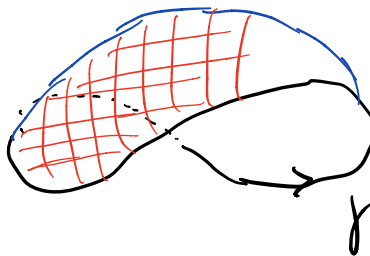
$$\vec{A}_{\text{dip}} = \frac{\mu_0 I}{4\pi} \cdot \frac{\vec{a} \times \vec{r}}{r}$$



cf. area A $\rightarrow \vec{a} = A \hat{n}$



$$\vec{a} = ?$$

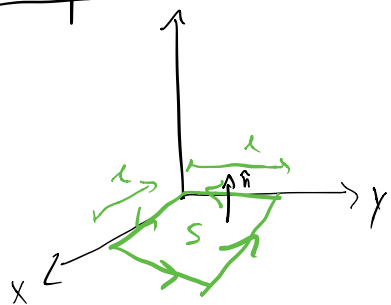


$$\vec{a} = \int_S \hat{n} da ; \oint \hat{n} da = 0$$

$\rightarrow$ área vetorial

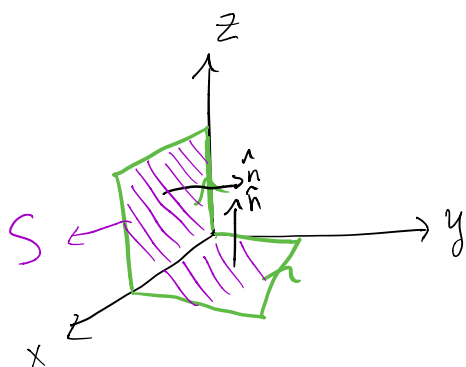
Momento do dipolo: $\vec{m} \equiv I \vec{a} \rightarrow \vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

Exemplo:



$$\vec{A} \simeq \frac{\mu_0}{4\pi} \frac{1}{r^2} \vec{m} \times \hat{r}$$

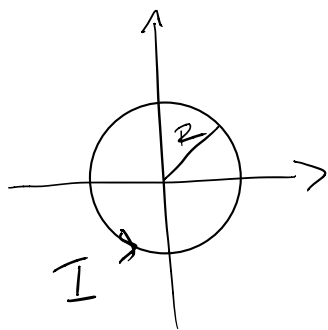
$$\vec{m} = I \vec{a} = I \int_S \hat{n} da = I \hat{z} \int_S da = \underline{I l^2 \hat{z}} \quad \underline{l \ll r}$$



$$\vec{a} = \int_S \hat{n} da = \int_{S_1} \hat{n} da + \int_{S_2} \hat{n} da$$

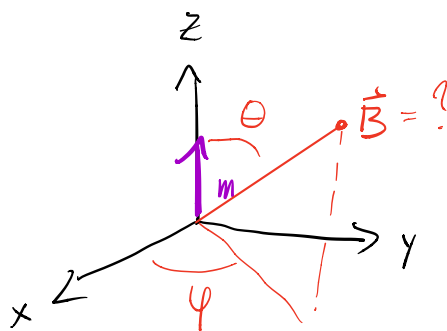
$$\vec{a} = \hat{z} \int_{S_1} da + \hat{y} \int_{S_2} da = l^2 (\hat{z} + \hat{y})$$

$$\vec{m} = I l^2 (\hat{z} + \hat{y})$$



$$\vec{m} = I \underbrace{\pi R^2}_{\vec{a}} \hat{z}$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^3}; \quad \vec{m} = m \hat{z}$$



Qual o campo magnético de um dipolo?

$$\vec{B} = \nabla \times \vec{A}$$

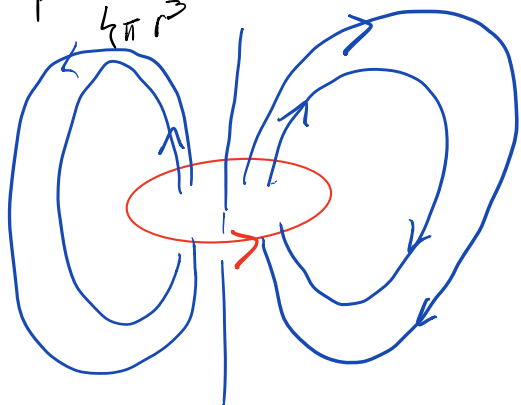
$$\vec{m} \times \vec{r} = (-my, mx, 0) = (-mr \sin\theta \sin\phi, mr \sin\theta \cos\phi, 0)$$

$$= mr \sin\theta (-\sin\phi, \cos\phi, 0) = mr \sin\theta \hat{\phi}$$

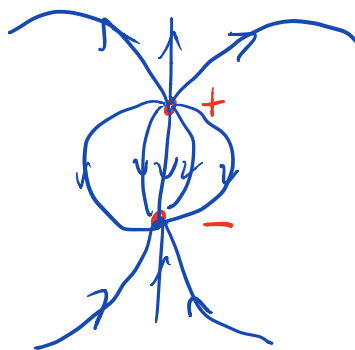
$$\vec{A} = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi}; \quad \vec{\nabla} \times \vec{A} = \frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta}) = \vec{B}_{\text{dip}}$$

R analogo ao campo elétrico

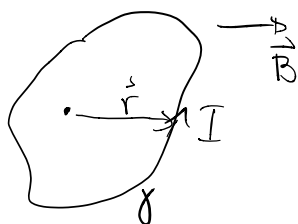
$$\vec{B}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$$



$$\vec{E}_{\text{dip}} = \frac{P}{4\pi\epsilon_0} \cdot \frac{1}{r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$$



$$\vec{F} = q \vec{v} \times \vec{B}$$



$$d\vec{F} = I d\vec{l} \times \vec{B} = -I \vec{B} \times d\vec{l}$$

$$\vec{F} = -I \oint_{\gamma} \vec{B} \times d\vec{l} \quad \text{se } \vec{B} \text{ for uniforme}$$

$$\vec{F} = -I \vec{B} \times \oint_{\gamma} d\vec{l} = \underline{0}$$

Torque $d\vec{N} = I \vec{r} \times d\vec{F} = I \vec{r} \times (d\vec{l} \times \vec{B})$

$$\vec{N} = I \oint \vec{r} \times (d\vec{l} \times \vec{B})$$